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# Stochastic nonhomogeneous incompressible Navier–Stokes equations

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**Abstract**

We construct solutions for 2- and 3-D stochastic nonhomogeneous incompressible Navier–Stokes equations with general multiplicative noise. These equations model the velocity of a mixture of incompressible fluids of varying density, influenced by random external forces that involve feedback; that is, *multiplicative* noise. Weak solutions for the corresponding deterministic equations were first found by Kazhikhov [A.V. Kazhikhov, Solvability of the initial and boundary-value problem for the equations of motion of an inhomogeneous viscous incompressible fluid, Soviet Phys. Dokl. 19 (6) (1974) 331–332; English translation of the paper in: Dokl. Akad. Nauk SSSR 216 (6) (1974) 1240–1243]. A stochastic version with *additive* noise was solved by Yashima [H.F. Yashima, Equations de Navier–Stokes stochastiques non homogènes et applications, Thesis, Scuola Normale Superiore, Pisa, 1992].

The methods here extend the Loeb space techniques used to obtain the first general solutions of the stochastic Navier–Stokes equations with multiplicative noise in the homogeneous case [M. Capiński, N.J. Cutland, Stochastic Navier–Stokes equations, Applicandae Math. 25 (1991) 59–85]. The solutions display more regularity in the 2D case. The methods also give a simpler proof of the basic existence result of Kazhikhov.


*Keywords:* Nonhomogeneous; Density-dependent; Stochastic Navier–Stokes equations; Loeb space

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## 1. Introduction

This paper is concerned with the stochastic nonhomogeneous (i.e., nonconstant density) incompressible Navier–Stokes equations with *multiplicative* noise:

$$\rho\, du = \big[\nu \Delta u - \langle \rho u, \nabla\rangle u - \nabla p + \rho f(t,u)\big]\, dt + \rho g(t,u)\, dw_t, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \langle u, \nabla\rangle \rho = 0, \tag{2}$$

$$\operatorname{div} u = 0, \qquad u|_{\partial D} = 0, \qquad u|_{t=0} = u_0 \quad \text{and} \quad \rho|_{t=0} = \rho_0.$$

These model the velocity $u$ and density $\rho$ of a mixture of viscous incompressible fluids of varying density in a bounded domain $D \subset \mathbb{R}^d$ ($d = 2, 3$). As usual $p$ is the pressure; $f$ represents external forces and the term $g\, dw$ (where $w$ is a Wiener process) represents additional random forces.

Setting $g = 0$ gives the deterministic nonhomogenous equations. For these, Kazhikhov [19] obtained weak solutions to (1), (2) in the conventional Hilbert space setting, assuming the initial density to be bounded away from zero; extra regularity was exhibited in dimension $d = 2$. For an exposition see the monograph [2]. Subsequently weak solutions were found for these and related equations assuming only that the initial density is non-negative [20,21,23,24]; the paper [22] further investigated the regularity of solutions. More recently results on the local existence of strong solutions have been obtained—see, for example, [4,9,10].

The *stochastic* equations with *additive* noise (where the term $dG = g\, dw$ does not depend on $u$) allow an essentially pathwise approach, and were solved by Yashima [27], who assumed positive initial density. The recent paper [3] purports to deal with a stochastic version of *compressible* Navier–Stokes but the equation considered lacks the crucial (and problematic) bilinear term $\langle \rho u, \nabla\rangle u$; moreover it also deals only with additive noise and is hence not truly stochastic. In some work that is distantly related, stochastic equations for a viscous gas have been treated by Tornatore [26] and Yashima [28], in both cases with additive noise. Tornatore works in 2 space dimensions and assumes periodic boundary conditions; Yashima works an infinite 1-dimensional setting.

In this paper we establish existence for the stochastic equations (1), (2) with general *multiplicative* noise in space dimensions $d \leqslant 3$, but still assuming that the density is bounded away from zero. The techniques are an extension of the Loeb space methods developed by the first author and Capiński [5–7] to establish general existence for the stochastic Navier–Stokes equations with constant density, and used more recently to construct attractors in 2 and 3 dimensions [8,15,16] and in optimal control theory for the stochastic equations [13,14]. In dimension 2 the solutions constructed are shown to have additional regularity.

When $g = 0$ the techniques here give a simplified proof of Kazhikov's original existence theorem.

The results in this paper are completely standard, although Loeb spaces and other nonstandard ideas are central to the proofs. In Appendix A we give a very brief outline of the basics of this theory as employed here; some particularly important results that are needed are noted in the preliminaries (Section 2.2).

The work reported here is an extension of results contained in the second author's PhD thesis [17] written under the supervision of the first author.

## 2. Preliminaries

### 2.1. The functional setting of the Navier–Stokes equations

We assume that the domain $D$ is an open bounded subset of $\mathbb{R}^d$ $(d = 2, 3)$ with boundary sufficiently smooth (see below). Write $\mathbf{L}^2(D) = L^2(D, \mathbb{R}^d)$ and write $H^m(D)$ for the classical Sobolev space $W^{m,2}(D)$.

The spaces central to the conventional Hilbert space formulation of the Navier–Stokes equation are $\mathbf{H}$ and $\mathbf{V}$ defined as follows. Let

$$\mathcal{V} = \{u \in \mathcal{C}_0^\infty(D, \mathbb{R}^d)\colon\ \operatorname{div} u = 0\}.$$

Then $\mathbf{H}$ is the closure of $\mathcal{V}$ in $\mathbf{L}^2(D)$ with the norm given by $|u|^2 = (u, u)$, where

$$(u, v) = \sum_{i=1}^{d} \int_D u^i(x) v^i(x)\, dx,$$

and $\mathbf{V}$ is the closure of $\mathcal{V}$ in the norm $|u| + \|u\|$, where $\|u\|^2 = ((u, u))$ and

$$((u, v)) = \sum_{j=1}^{d} \left( \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right).$$

$\mathbf{H}$ and $\mathbf{V}$ are real Hilbert spaces, $\mathbf{V}$ dense in $\mathbf{H}$. The dual space to $\mathbf{V}$ is denoted by $\mathbf{V}'$ with the duality extending the scalar product in $\mathbf{H}$ and

$$\mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}'.$$

Write $A$ for the Stokes operator on $\mathbf{H}$ (the self-adjoint extension of the projection of $-\Delta$) which is densely defined in $\mathbf{H}$; it is extended to $A : \mathbf{V} \to \mathbf{V}'$ by $Au[v] = ((u, v))$ for $u, v \in \mathbf{V}$. The operator $A$ has an orthonormal basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}} \subset \mathbf{H}$ with eigenvalues $0 < \lambda_k \nearrow \infty$. For $u \in \mathbf{H}$ write $u = \sum u_k e_k$. Write $\mathbf{H}_n$ for the finite-dimensional subspace $\mathbf{H}_n = \operatorname{span}\{e_1, e_2, \dots, e_n\}$ and $\mathrm{Pr}_n$ for the projection onto $\mathbf{H}_n$. We assume the boundary of $D$ is smooth enough to ensure that each $e_n \in C^1(D)$, for example, $\partial D$ is of class $C^2$.

A family of spaces $\mathbf{H}^r$ for $r \in \mathbb{R}$ is defined as follows: for $r \geqslant 0$

$$\mathbf{H}^r = \left\{ u \in \mathbf{H}\colon \sum_{k=1}^{\infty} \lambda_k^r u_k^2 < \infty \right\}$$

with the norm given by

$$|u|_r^2 = \sum_{k=1}^{\infty} \lambda_k^r u_k^2$$

and $\mathbf{H}^{-r}$ is the dual of $\mathbf{H}^r$. We may represent $\mathbf{H}^{-r}$ by

$$\mathbf{H}^{-r} = \left\{ (u_k)_{k\in\mathbb{N}}\colon \sum_{k=1}^{\infty} \lambda_k^{-r} u_k^2 < \infty \right\}.$$

In terms of this family we have $\mathbf{H}^0 = \mathbf{H}$, $\mathbf{H}^1 = \mathbf{V}$, and $\mathbf{H}^{-1} = \mathbf{V}'$ with the norms $|u| = |u|_0$ and $\|u\| = |u|_1$.

The trilinear form $b(u, v, y)$ is defined for $u, v, y \in \mathbf{L}^2(D)$ by

$$b(u, v, y) = \sum_{i,j=1}^{d} \int_D u^i(x) \frac{\partial v^j}{\partial x_i}(x) y^j(x)\, dx = \big(\langle u, \nabla\rangle v, y\big)$$

whenever the integral makes sense.[1] It has many properties—see [25], for example. We will often encounter terms of the form $b(\theta u, v, y)$ where $\theta \in L^\infty(D)$ and $u, v, y \in \mathbf{V}$, and the properties of these that are needed are noted below (these slightly generalize those normally encountered).

Also noted below are some properties of a second trilinear form $\beta$ occurring in the development,

$$\beta(\theta, u, \varphi) = \sum_{i=1}^{d} \int_D \theta(x) u^i(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx = (\theta u, \nabla\varphi)$$

which is defined, for example, for $\theta \in L^\infty(D)$, $u \in \mathbf{L}^2(D)$ and $\varphi \in H^1(D)$.

**Lemma 2.1.** *When all terms are defined*

(a) *if $u, v, y \in \mathbf{V}$ and $\theta \in L^\infty(D) \cap H^1(D)$ then*

$$\big(u\langle v, \nabla\rangle\theta, y\big) = -b(\theta v, u, y) - b(\theta v, y, u); \tag{3}$$

(b) *if $u, v, y \in \mathbf{V}$ and $\theta \in L^\infty(D)$ then*

$$\big|b(\theta u, v, y)\big| \leqslant c|\theta|\, |u|^{\frac14} \|u\|^{\frac34} \|v\| |y|^{\frac14} \|y\|^{\frac34}, \tag{4}$$

$$\big|b(\theta u, v, y)\big| \leqslant c|\theta| \|u\| \|v\| \|y\|; \tag{5}$$

(c) *if $d = 2$ and $u \in \mathbf{V}$, $v \in \mathbf{H}_2$, $y \in \mathbf{H}$ and $\theta \in L^\infty(D)$ then*

$$\big|b(\theta u, v, y)\big| \leqslant c|\theta| |u|^{\frac12} \|u\|^{\frac12} \|v\|^{\frac12} |Av|^{\frac12} |y|; \tag{6}$$

(d) *if $\theta \in L^\infty(D) \cap H^1(D)$, $u \in \mathbf{L}^2(D)$ and $\varphi \in H^1(D)$ then*

$$\beta(\theta, u, \varphi) + \beta(\varphi, u, \theta) = 0;$$

[1] Of course, $u, v, y \in \mathbf{L}^2(D)$ is not *sufficient* for $b(u, v, y)$ to be defined—extra regularity is required.

(e) *if* $\theta \in L^\infty(D)$, $u \in \mathbf{L}^2(D)$ *and* $\varphi \in H^1(D)$

$$\left|\beta(\theta, u, \varphi)\right| \leqslant c|\theta||u||\nabla\varphi|. \tag{7}$$

**Proof.** All routine extensions of well known results. □

When $\theta \equiv 1$ identity (3) gives the well-known relations $b(v, u, y) + b(v, y, u) = 0$ and $b(v, u, u) = 0$; and the familiar continuity properties of $b$ follow from (4), (5).

Inequality (4) gives the second of the following inequalities.

**Proposition 2.2.** *For* $u \in \mathbf{V}$

$$|Au|_{V'} = \|u\|, \qquad \left|B(u)\right|_{V'} \leqslant c|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}},$$

*where* $B(u) = b(u, u, \cdot)$.

If $y \in \mathbf{L}^2(D)$ and $\theta \in L^\infty(D)$ we will often encounter the function $\theta y \in \mathbf{L}^2(D)$. For the weak solutions that will be considered the projection onto $\mathbf{V}'$ will occur. To formalize this, we make the following definition.

**Definition 2.3.** For any subspace $\mathbf{X}$ of $\mathbf{L}^2(D)$ define an operator $\theta_\mathbf{X} : \mathbf{L}^2(D) \to \mathbf{X}$ by

$$\theta_\mathbf{X} y = \mathrm{Pr}_\mathbf{X}(\theta y).$$

This means that

$$(\theta y, v) = (\theta_\mathbf{X} y, v)$$

for all $v \in \mathbf{X}$. When $\mathbf{X} = \mathbf{H}_n$ simply write $\theta_n$ for $\theta_{\mathbf{H}_n}$.

The following observation will be important in the solution of finite-dimensional approximations to the equations.

**Lemma 2.4.** *Let* $\theta \in L^\infty(D)$ *with* $0 < m \leqslant \theta(x) \leqslant M$ *for all* $x$ *and let* $\tilde{\theta}_n$ *be the restriction of* $\theta_n$ *to* $\mathbf{H}_n$. *Then*

(a) $|\theta_n| \leqslant M$,
(b) $\tilde{\theta}_n$ *is self-adjoint and positive definite, hence invertible. Moreover* $|\tilde{\theta}_n| \leqslant M$ *and* $|\tilde{\theta}_n^{-1}| \leqslant m^{-1}$.

**Proof.** (a) Write $\theta y = \theta_n y + w$, where $w \in \mathbf{H}_n^\perp$. Then

$$|\theta_n y|^2 \leqslant |\theta y|^2 \leqslant M^2|y|^2,$$

so $|\theta_n| \leqslant M$.

(b) For $u, v \in \mathbf{H}_n$

$$(\tilde{\theta}_n u, v) = \int_D \bigl\langle \theta(x)u(x), v(x)\bigr\rangle\,dx = \int_D \bigl\langle u(x), \theta(x)v(x)\bigr\rangle\,dx = (u, \tilde{\theta}_n v) \quad \text{and}$$

$$(\tilde{\theta}_n u, u) = \int_D \theta(x)\bigl|u(x)\bigr|^2\,dx,$$

so

$$m|u|^2 \leqslant (\tilde{\theta}_n u, u) \leqslant M|u|^2.$$

Invertibility and the estimate on the norms follow from elementary linear algebra since $\mathbf{H}_n \cong \mathbb{R}^n$. □

We need one further preliminary result: for any given function $Y:[0,T] \to \mathbf{H}_n$ there is a solution to the density equation. This is shown in the following lemma.

**Lemma 2.5.** *If $Y \in C(0,T;\mathbf{H}_n)$ and $\rho_0 \in C^1(D)$ with*

$$0 < m \leqslant \rho_0(x) \leqslant M$$

*then the equation*

$$\frac{\partial\rho}{\partial t} + \langle Y, \nabla\rangle\rho = 0, \qquad \rho(0) = \rho_0 \tag{8}$$

*has a unique solution $\rho \in C^1([0,T]\times D)$. The solution has*

$$0 < m \leqslant \rho(t,x) \leqslant M$$

*for all $(t,x)$. The dependence of $\rho$ on $Y$ is continuous, in the sense that if $r(Y)$ denotes the solution to the density equation* (8)*, so that*

$$r : C(0,T;\mathbf{H}_n) \to C^1\bigl([0,T]\times D\bigr)$$

*then $r$ is continuous with respect to the uniform topologies on both sides.*

**Proof.** The proof is routine using the method of characteristics. □

### 2.2. Nonstandard analysis

The basics of nonstandard analysis and Loeb theory are assumed—these can be found in [1,7,11] or [12], for example. In particular, we work in a standard universe $\mathbb{V} = \mathbb{V}(S)$, where $S$ is a base set that contains all the objects of interest, and take an $\aleph_1$-saturated extension ${}^*\mathbb{V} \subset \mathbb{V}({}^*S)$. For ease of reference we gather together in Appendix A the basic facts including the representation of the spaces occurring in the study of the NSE equations. Here we note some particularly important results that will be needed.

For $L^\infty(D)$ the relevant topology is the weak* topology (bearing in mind that $L^\infty(D) = (L^1(D))'$). The Banach–Alaoglu theorem tells us that the unit ball in $L^\infty(D)$ is compact in this topology, so we have the following:

**Proposition 2.6.** *Let $\Theta \in {}^*L^\infty(D)$ with $|\Theta|$ finite. Then $\Theta$ is near-standard in the weak* topology. If we denote the standard part by $\widehat{\Theta}$ (to avoid confusion with other standard parts) then*

(i) $\int_{^*D} \Theta\, {}^*z\, d\xi \approx \int_D \widehat{\Theta} z\, dx$ *for all* $z \in L^1(D)$;
(ii) $|\widehat{\Theta}| \leqslant |\Theta|$;
(iii) *if* $\Theta(\xi) \geqslant m \geqslant 0$ *for all* $\xi \in {}^*D$ *then* $\widehat{\Theta}(x) \geqslant m$ *for all* $x \in D$;
(iv) $\Theta \approx \widehat{\Theta}$ *in* $L^p(D)$ *weakly for* $1 < p < \infty$.

**Proof.** All routine. □

Although we will not need it, the standard part $\widehat{\Theta}$ can defined explicitly by

$$\widehat{\Theta}({}^\circ\xi) = \mathbb{E}({}^\circ\Theta|\mathcal{G})(\xi),$$

where $\mathcal{G}$ is defined from the Borel algebra $\mathcal{B} = \mathcal{B}(D)$ by $\mathcal{G} = \mathrm{st}^{-1}(\mathcal{B})$.

Any easy consequence of Proposition 2.6 is:

**Lemma 2.7.** *Let $\Theta \in {}^*L^\infty(D)$ with $|\Theta|$ finite. Then for any $z \in L^1(D)$ with S-integrable lifting $Z: {}^*D \to {}^*\mathbb{R}$*

$$\int_{^*D} \Theta(\xi) Z(\xi)\, d\xi \approx \int_D \widehat{\Theta} z\, dx.$$

**Proof.** For Loeb-a.a. $\xi$ we have $\Theta(\xi)Z(\xi) \approx \Theta(\xi)\,{}^*z(\xi)$ by Anderson's theorem. Since $Z$ is S-integrable so is $\Theta Z$; hence

$$\int_{^*D} \Theta(\xi) Z(\xi)\, d\xi \approx \int_{^*D} \Theta(\xi)\, {}^*z(\xi)\, d\xi \approx \int_D \widehat{\Theta} z\, dx$$

using Proposition 2.6(i). □

From this we have the following; part (b) is the counterpart of the so-called "Crucial lemma" of [7].

**Corollary 2.8.** *Let $\Theta \in {}^*L^\infty(D)$ with $|\Theta|$ finite. If $U, V \in {}^*\mathbf{H}$ and $U \approx u$ and $V \approx v$ in $\mathbf{H}$ and $\theta = \widehat{\Theta}$ then*

(a) $$(\Theta U, V) \approx (\theta u, v); \tag{9}$$
(b) *if $U, V \in {}^*\mathbf{V}$ with $\|U\|, \|V\|$ finite and $w \in \mathbf{V}$ then*

$${}^*b(\Theta U, {}^*w, V) \approx b(\theta u, w, v). \tag{10}$$

**Proof.** (a) It is sufficient to observe that $\langle U(\xi), V(\xi)\rangle$ is an S-integrable lifting of $\langle u(x), v(x)\rangle \in L^1(D)$. This follows because $U \approx u$ in $\mathbf{H}$ is equivalent to $U$ being an $\mathrm{SL}^2$ lifting of $u$.

(b) Using the continuity property (4) together with $|U - {}^*u| \approx 0 \approx |V - {}^*v|$ and the fact that $\|U\|, \|V\|$ are finite we have

$$ {}^*b(\Theta U, {}^*w, V) \approx {}^*b(\Theta\, {}^*u, {}^*w, {}^*v). $$

Now observe that ${}^*b(\Theta {}^*u, {}^*w, {}^*v) = \int_{{}^*D} \Theta\, {}^*z$, where $z = \langle v, \langle u, \nabla\rangle w\rangle \in L^1(D)$, so

$$ {}^*b(\Theta\, {}^*u, {}^*w, {}^*v) \approx \int_D \widehat{\Theta} z = b(\widehat{\Theta} u, w, v), $$

which gives (10). $\square$

## 3. Stochastic nonhomogeneous Navier–Stokes equations

The general stochastic version of the nonhomogeneous incompressible Navier–Stokes equations that we solve below is

$$ \rho\, du = \big[\nu \Delta u - \langle \rho u, \nabla\rangle u - \nabla p + \rho f(t,u)\big]\, dt + \rho g(t,u)\, dw_t, \tag{11} $$

$$ \frac{\partial \rho}{\partial t} + \langle u, \nabla\rangle \rho = 0, \tag{12} $$

$$ \operatorname{div} u = 0, \tag{13} $$

obtained from the corresponding deterministic equation by adding a random force. The driving random process $w$ is taken to be a Wiener process in $\mathbf{H}$ with covariance $Q$ of trace class, and the stochastic integral is the extension of the Itô integral to Hilbert spaces due to Ichikawa [18], as in [7], defined for the present context in the definition of a solution below. As already noted we work on a bounded domain $D \subset \mathbb{R}^d$.

### 3.1. Definition of solution

The definition of a weak solution to the stochastic equations is the natural generalization of that employed by Kazhikhov [19] for the case $g = 0$. Both the velocity and the density will be stochastic processes living on an adapted probability space $\boldsymbol{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$.

**Definition 3.1.** Given $u_0 \in \mathbf{H}$, $\rho_0 \in L^\infty(D)$, $f : [0,T] \times \mathbf{H} \to \mathbf{H}$ and $g : [0,T] \times \mathbf{H} \to L(\mathbf{H}, \mathbf{H})$ a pair of stochastic processes $(\rho, u)$ is a *weak solution* to the stochastic nonhomogeneous Navier–Stokes equations if

(i) $u \in L^2([0,T] \times \Omega, \mathbf{V})$ and for a.a. $\omega$

$$ u(\cdot, \omega) \in L^\infty(0,T; \mathbf{H}) \cap L^2(0,T; \mathbf{V}); $$

(ii) $\rho \in L^\infty([0,T] \times D \times \Omega)$;

(iii) for almost all $T_0 \leqslant T$, for all $\Phi \in C^1(0,T;\mathbf{V})$

$$\big(\rho(T_0)u(T_0),\Phi(T_0)\big)-\big(\rho_0u_0,\Phi(0)\big) = \int_0^{T_0}\big[\big(\rho u,\Phi'+\langle u,\nabla\rangle\Phi\big)-\nu((u,\Phi))+(\rho f,\Phi)\big]\,dt+\int_0^{T_0}(\Phi,\rho g)\,dw; \tag{14}$$

(iv) for all $\varphi \in C^1(0,T;H^1(D))$, for all $T_0 \leqslant T$

$$\big(\rho(T_0),\varphi(T_0)\big)-\big(\rho_0,\varphi(0)\big)=\int_0^{T_0}\big(\rho,\varphi'+\langle u,\nabla\rangle\varphi\big)\,dt; \tag{15}$$

(v) $\rho(0)=\rho_0$ and $u(0)=u_0$.

When $g=0$ this gives the well-known definition of a weak solution for the deterministic equations as in [19].

The conditions under which we solve Eqs. (11)–(13) are the following:

**Conditions 3.2.** (a) $f:[0,T]\times\mathbf{H}\to\mathbf{H}$ and $g:[0,T]\times H\to L(H,H)$ are jointly measurable and there is $a\in L^2[0,T]$ and $0<m\leqslant M$ such that

(i) $f(t,\cdot)\in C(K_r,\mathbf{H})$ for each $r$, where $K_r=\{u\colon \|u\|\leqslant r\}$ with the strong topology of $\mathbf{H}$;
(ii) $g(t,\cdot)\in C(K_r,L(\mathbf{H},\mathbf{H}))$ for each $r$;
(iii) $|f(t,u)|+|g(t,u)|_{\mathbf{H},\mathbf{H}}\leqslant a(t)(1+|u|)$ for all $u\in\mathbf{H}$.

(b) $\rho_0\in L^\infty(D)$ with $m\leqslant\rho_0(x)\leqslant M$ for all $x$.

The procedure for constructing a solution is an extension of that developed in [7] for the homogeneous stochastic equations: use a Loeb space as the underlying probability space, solve a modified version of the Galerkin approximation of dimension $N$ (infinite) and then take standard parts.

### 3.2. *The adapted Loeb space*

As in [7] we take $\boldsymbol{\Omega}$ to be an adapted Loeb space[2] defined as follows. Fix an infinite $N$ and take a nonstandard (internal) filtered probability space

$$\boldsymbol{\Omega}_0=\big(\Omega,\mathcal{A},(\mathcal{A}_\tau)_{\tau\geqslant 0},\Pi\big)$$

carrying an internal Wiener process $W(\tau)\in\mathbf{H}_N$ with covariance $Q_N=\mathrm{Pr_N}\,{}^*Q\restriction\mathbf{H}_N$, adapted to $(\mathcal{A}_\tau)_{\tau\geqslant 0}$. (A canonical candidate is provided by taking $\Omega={}^*C(0,\infty;\mathbf{H}_N)$ and the appropriate

2 See Appendix A for a brief introduction to Loeb measure theory.

Wiener measure—see [7] for details.) Writing $\Pi_L = P$ for the Loeb measure we then can define the adapted (or filtered) Loeb space

$$\boldsymbol{\Omega} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P\big),$$

where $\mathcal{F} = L(\mathcal{A})$ and the right continuous filtration $(\mathcal{F}_t)$ is defined by

$$\mathcal{F}_t = \bigcap_{t<{}^\circ\tau} \sigma(\mathcal{A}_\tau) \vee \mathcal{N}$$

(here $\mathcal{N}$ denotes the family of $P$-null sets). It is shown in [7] that a.a. the internal process $W(\tau)$ is near-standard in $\mathbf{H}$ and S-continuous and so

$$w = {}^\circ W$$

defines a Wiener process on $\boldsymbol{\Omega}$ of covariance $Q$. From now on we fix this adapted space and the Wiener process $w$.

### 3.3. Hyperfinite approximation of dimension N

The first step towards solving (11)–(13) is to formulate and solve a modified version of the Galerkin approximation of dimension $N$ on the internal space $\boldsymbol{\Omega}_0$. The solution will be a pair of internal stochastic processes $(R, U)$ with $R : {}^*[0,T] \times \Omega \to {}^*C^1(D)$ and $U : {}^*[0,T] \times \Omega \to \mathbf{H}_N$.

As (unmodified) Galerkin approximation of dimension $N$ take the following equations for such a pair $(R, U)$. For each fixed $\tau$ we have $R(\tau) \in {}^*C^1(D) \subset {}^*L^\infty(D)$ so we have the operator $R_N(\tau) : {}^*\mathbf{L}^2(D) \to \mathbf{H}_N$ (see Definition 2.3). Write $G = \mathrm{Pr_N}\, {}^*g \restriction \mathbf{H}_N$ and $F = \mathrm{Pr_N}\, {}^*f$; then the equations are

$$\begin{aligned} R_N(\tau)\, dU(\tau) = {} & \big[-R_N(\tau)\big\langle U(\tau), \nabla\big\rangle U(\tau) - \nu A U(\tau) + R_N(\tau) F\big(\tau, U(\tau)\big)\big]\, d\tau \\ & + R_N(\tau) G\big(\tau, U(\tau)\big)\, dW_\tau \end{aligned} \tag{16}$$

(an equation in $\mathbf{H}_N$), and for $^*$a.a. $\omega$

$$\frac{dR}{d\tau} + \big\langle U(\tau), \nabla\big\rangle R(\tau) = 0 \tag{17}$$

with prescribed initial conditions $U(0) = U_0 \in \mathbf{H}_N$ and $R(0) = R_0 \in {}^*C^1(D)$. It is necessary to modify these equations in order to take care of the quadratic term, as follows.

Fix an infinite number $\kappa$ and for $V \in \mathbf{H}_N$ define the truncation $\overline{V}$ by

$$\overline{V} = \begin{cases} V & \text{if } |V| \leqslant \kappa, \\ \kappa V/|V| & \text{if } |V| \geqslant \kappa. \end{cases}$$

The modified equations are then

$$\begin{aligned} R_N(\tau)\, dU(\tau) = {} & \big[-R_N(\tau)\big\langle \overline{U}(\tau), \nabla\big\rangle U(\tau) - \nu A U(\tau) + R_N(\tau) F\big(\tau, U(\tau)\big)\big]\, d\tau \\ & + R_N(\tau) G\big(\tau, U(\tau)\big)\, dW_\tau \end{aligned} \tag{18}$$

and for ${}^*$a.a. $\omega$

$$\frac{dR}{d\tau} + \langle \overline{U}(\tau), \nabla \rangle R(\tau) = 0 \tag{19}$$

with initial conditions $U(0) = U_0 \in \mathbf{H}_N$ and $R(0) = R_0$. For these we have:

**Theorem 3.3.** *Assume Conditions* 3.2 *on the coefficients $f, g$. If $U_0 \in \mathbf{H}_N$ is finite and $R_0 \in {}^*C^1(D)$ with $0 < m \leqslant R_0(\xi) \leqslant M$ then the internal modified equations* (18), (19) *have an internal solution with the following properties*:

(a) *There is a finite constant $E$ (independent of $N$) such that*

$$\mathbb{E}\left( \sup_{\tau \leqslant T} \left|U(\tau)\right|^2 + \nu \int_0^T \left\|U(\sigma)\right\|^2 d\sigma \right) < E. \tag{20}$$

(b) *For ${}^*$a.a. $\omega$*

$$m \leqslant R(\tau, \xi, \omega) \leqslant M$$

*for all $\tau$ and $\xi$.*

**Proof.** Recall the function $r$ of Lemma 2.5. For any $Y$ write $r(Y, t) = r(Y)(t, \cdot) \in C^1(D) \subset L^\infty(D)$ so that Definition 2.3 and Lemma 2.4 gives the operators ${}^*r_N(Y, \tau)$ and ${}^*\tilde{r}_N(Y, \tau)$ in the setting of $\mathbf{H}_N$. Then Eqs. (18), (19) can be written as the single equation

$$\begin{aligned} dU = {} & \tilde{r}(\overline{U}, \tau)_N^{-1}\left[-r(\overline{U}, \tau)_N \langle \overline{U}(\tau), \nabla \rangle U(\tau) - \nu A U(\tau) + r(\overline{U}, \tau)_N F\big(\tau, U(\tau)\big)\right] d\tau \\ & + G\big(\tau, U(\tau)\big)\, dW_\tau \end{aligned} \tag{21}$$

(where we have dropped the ${}^*$ on ${}^*r$ and ${}^*\tilde{r}$). This is an equation in $\mathbf{H}_N$ of the form

$$dU = h(\tau, U)\, d\tau + G\big(\tau, U(\tau)\big)\, dW_\tau,$$

where $h(\tau, U)$ depends on the past $U \restriction [0, t]$. These functions are jointly measurable, ${}^*$continuous and have linear growth; the boundedness of the operators $\tilde{r}(\overline{U}, t)_n^{-1}$ and $r(\overline{U}, t)_n$ and the growth conditions on $f, g$ give:

$$\left|h(t, U)\right| + \left|G\big(\tau, U(\tau)\big)\right|_{\mathbf{H}_N, \mathbf{H}_N} \leqslant C\big({}^*a(t) + 1\big)\big(1 + \left|U(t)\right|\big)$$

for some ${}^*$finite constant $C$ ($C$ will be infinite). The transfer of finite-dimensional SDE theory gives the existence of a solution $U(\tau, \omega)$ to (21) (which may not be unique unless $f$ and $g$ are suitably Lipschitz). Then putting $R = r(\overline{U}, \tau)$ the pair $(R, U)$ solves Eqs. (18), (19).

It is clear from the property of the function $r$ that Theorem 3.3(b) is satisfied, so it remains to show that $U$ satisfies the stochastic energy inequality (20). First, Itô's lemma gives:

$$\begin{aligned}
d(RU,U) &= \left[\left(\frac{dR}{d\tau}U,U\right) + 2(R\,dU,U) + \operatorname{tr}\big(Q_N R_N G^T G\big)\right]d\tau + 2(U,R_N G)\,dW_\tau \\
&= \big[-\big(U\langle\overline{U},\nabla\rangle R,U\big) + 2\big(\big[-R\langle\overline{U},\nabla\rangle U - \nu AU + RF(t,U)\big],U\big)\big]\,d\tau \\
&\quad + \operatorname{tr}\big(Q_N R_N G^T G\big)\,d\tau + 2(U,R_N G)\,dW_\tau \\
&= \big[-2\nu\|U\|^2 + 2\big(RF(t,U),U\big) + \operatorname{tr}\big(Q_N R_N G^T G\big)\big]\,d\tau + 2(U,R_N G)\,dW_\tau
\end{aligned}$$

since $(U\langle\overline{U},\nabla\rangle R,U) = -2b(R\overline{U},U,U)$ by (3) and by definition $(R\langle\overline{U},\nabla\rangle U,U) = b(R\overline{U},U,U)$. Young's inequality and the growth condition on $f$ give

$$\begin{aligned}
2\big(RF\big(\tau,U(\tau)\big),U(\tau)\big) &\leqslant \frac{M^2\lambda_1}{\nu}\big|F\big(\tau,U(\tau)\big)\big|^2 + \frac{\nu}{\lambda_1}\big|U(\tau)\big|^2 \\
&\leqslant \frac{M^2\lambda_1}{\nu}\big|F\big(\tau,U(\tau)\big)\big|^2 + \nu\big\|U(\tau)\big\|^2 \\
&\leqslant \nu\big\|U(\tau)\big\|^2 + \frac{M^2\lambda_1}{\nu}{}^*a^2(\tau)\big(1+\big|U(\tau)\big|\big)^2
\end{aligned}$$

and from the growth condition on $g$ we have

$$\begin{aligned}
\operatorname{tr}\big(Q_N R_N G^T G\big)(\tau) &= \operatorname{tr}\big(Q_N\big(R_N^{1/2}G\big)^T R_N^{1/2}G\big)(\tau) \leqslant \operatorname{tr} Q_N\cdot\big|R_N^{1/2}G\big|^2(\tau) \\
&\leqslant M\operatorname{tr} Q.{}^*a^2(\tau)\big(1+\big|U(\tau)\big|\big)^2.
\end{aligned}$$

Thus there is a finite constant $c_1$ such that for any $\tau \leqslant T$

$$\begin{aligned}
m\sup_{\sigma\leqslant\tau}\big|U(\sigma)\big|^2 &\leqslant \sup_{\sigma\leqslant\tau}\big(R(\sigma)U(\sigma),U(\sigma)\big) \\
&\leqslant M\big|U(0)\big|^2 - 2\nu\int_0^\tau \big\|U(\sigma)\big\|^2\,d\sigma + c_1\int_0^\tau {}^*a^2(\sigma)\big(1+\big|U(\sigma)\big|^2\big)\,d\sigma \\
&\quad + 2\sup_{\sigma\leqslant\tau}\big|I(\sigma)\big|,
\end{aligned}$$

where $I(\tau)$ is the internal martingale

$$I(\tau) = \int_0^\tau \big(U(\sigma),R_N(\sigma)G\big(\sigma,U(\sigma)\big)\big)\,dW_\sigma.$$

The argument in the proof of existence for the homogeneous stochastic Navier–Stokes equations (see [6] or [7, Theorem 6.4.1]) can now be replicated with $R_N G$ in place of $G$ to give the finite constant $E$ as in (a) of the theorem. □

A solution to the stochastic nonhomogeneous Navier–Stokes equations will be obtained by taking standard parts of the internal pair $(R,U)$ solving the modified equations (18), (19). Note that it follows from the energy bound (20) that for a.a. $\omega$ (with respect to $P$, the Loeb measure) $|U(\tau,\omega)|$ is finite and so $\overline{U}(\tau,\omega) = U(\tau,\omega)$ for all $\tau$.

Before taking standard parts we need two further properties of the pair $(R, U)$.

**Lemma 3.4.** *For almost all $\omega$ the function $R_N(\tau)U(\tau)$ is weakly S-continuous*; *that is, if $\sigma,\tau \in {}^*[0,T]$ with $\sigma \approx \tau$ then $R_N(\sigma)U(\sigma) \approx R_N(\tau)U(\tau)$ weakly in* $\mathbf{H}$.

**Proof.** For any $v \in \mathbf{V}$ we have, writing $U = U(\tau)$ and $R = R(\tau)$ and dropping $^*$ on $^*b$ to ease the notation:

$$\begin{aligned}
d(R_N U, {}^*v) = d(RU, {}^*v) &= \left(\frac{dR}{d\tau}U, {}^*v\right)d\tau + (R\,dU, {}^*v)\\
&= \big[-\big(U\langle\overline{U},\nabla\rangle R, {}^*v\big) - b(R\overline{U}, U, {}^*v) - \nu((U, {}^*v)) + \big(RF(\tau,U), {}^*v\big)\big]\,d\tau\\
&\quad + ({}^*v, R_N G\,dW_\tau)\\
&= b(R\overline{U}, {}^*v, U) - \nu((U, {}^*v)) + \big(RF(\tau,U), {}^*v\big) + ({}^*v, R_N G\,dW_\tau)
\end{aligned}$$

so

$$\begin{aligned}
&\big(R_N(\sigma)U(\sigma), {}^*v\big) - \big(R_N(0)U(0), {}^*v\big)\\
&\quad = \int_0^\sigma b(R\overline{U}, {}^*v, U)\,d\tau - \int_0^\sigma \nu((U, {}^*v))\,d\tau + \int_0^\sigma \big(R_N(\tau)\,{}^*f(\tau,U), {}^*v\big)\,d\tau\\
&\qquad + \int_0^\sigma ({}^*v, R_N G\,dW_\tau).
\end{aligned} \tag{22}$$

To establish S-continuity almost surely, consider each of the integrals on the right in turn. We have, as noted above, for almost all $\omega$, $\overline{U}(\sigma) = U(\sigma)$ for all $\sigma$, so

$$\int_0^T \big|b(R\overline{U}, {}^*v, U)\big|^{\frac{4}{3}}\,d\tau \leqslant c\int_0^T M|U|^{\frac{2}{3}}\|U\|^2\|{}^*v\|^{\frac{4}{3}} < \infty$$

using (4) and the energy estimate (20). For the next two integrals we have, almost surely

$$\int_0^T ((U, {}^*v))^2\,d\tau \leqslant \int_0^T \|U\|^2\|{}^*v\|^2\,d\tau < \infty \quad \text{and}$$

$$\int_0^T \big|\big(R_N(\tau)F(\tau,U), {}^*v\big)\big|^2\,d\tau \leqslant M^2\int_0^T {}^*a^2(\tau)\big(1+\big|U(\tau)\big|\big)^2|{}^*v|^2\,d\tau < \infty.$$

So by Lindstrøm's lemma the first three integrands on the right of (22) are S-integrable, and so the corresponding integrals are S-continuous, almost surely.

For the remaining term we simply invoke the nonstandard theory of stochastic integration as in [7, Section 3.6]; using (20) again and the growth condition on $g$ gives

$$\mathbb{E}\left(\int_0^T |R_N G|^2\right) d\tau < \infty$$

and so the integral $\int_0^\tau R_N G\, dW_\tau$ is a.s. S-continuous in $\mathbf{H}_N$; consequently $\int_0^\tau ({}^*v, R_N G\, dW_\tau)$ is a.s. S-continuous.

Thus, for almost all $\omega$, whenever $\sigma_1 \approx \sigma_2$ then $(R_N(\sigma_1)U(\sigma_1), {}^*v) \approx (R_N(\sigma_2)U(\sigma_2), {}^*v)$. Applying this to ${}^*v = {}^*e_k$ for each finite $k$ establishes that, almost surely, whenever $\sigma_1 \approx \sigma_2$ then $R_N(\sigma_1)U(\sigma_1) \approx R_N(\sigma_2)U(\sigma_2)$ weakly in $\mathbf{H}$. □

The second result is:

**Lemma 3.5.** *For almost all $\omega$, whenever $\|U(\sigma)\|$, $\|U(\tau)\|$ are finite and $\sigma \approx \tau$ then $U(\sigma) \approx U(\tau)$ strongly in* $\mathbf{H}$.

**Proof.** First, note that almost surely (with respect to the Loeb measure)

(i) $\sup_{\tau \leqslant T} |U(\tau)|^2 + \nu \int_0^T \|U(\tau)\|^2\, d\tau$ is finite and so $\overline{U}(\tau) = U(\tau)$ for all $\tau$,
(ii) $R_N(\tau)U(\tau)$ is weakly S-continuous,
(iii) $R$ satisfies the density equation $\frac{dR}{d\tau} + \langle \overline{U}(\tau), \nabla\rangle R(\tau) = 0$.

Now fix an $\omega$ for which (i)–(iii) holds and (suppressing $\omega$) take $\sigma \approx \tau$ with $\|U(\sigma)\|$, $\|U(\tau)\|$ finite.

Since $m|U(\tau) - U(\sigma)|^2 \leqslant (R(\sigma)(U(\tau) - U(\sigma)), U(\tau) - U(\sigma))$ it is sufficient to show that

$$\big(R(\sigma)\big(U(\tau) - U(\sigma)\big), U(\tau) - U(\sigma)\big) \approx 0.$$

Now

$$R(\sigma)\big(U(\tau) - U(\sigma)\big) = \big[R(\tau)U(\tau) - R(\sigma)U(\sigma)\big] - \big(R(\tau) - R(\sigma)\big)U(\tau),$$

so it suffices to prove

$$\big(R(\tau)U(\tau) - R(\sigma)U(\sigma), V\big) \approx 0 \quad \text{and} \tag{23}$$

$$\big(\big(R(\tau) - R(\sigma)\big)U(\tau), V\big) \approx 0 \tag{24}$$

for $V = U(\tau) - U(\sigma)$, which is strongly nearstandard in $\mathbf{H}$ since $\|V\|$ is finite.

For (23) we have from the previous lemma that $R(\sigma)U(\sigma) \approx R(\tau)U(\tau)$ weakly in $\mathbf{H}$, so

$$0 \approx \big(R(\tau)U(\tau) - R(\sigma)U(\sigma), {}^\circ V\big) \approx \big(R(\tau)U(\tau) - R(\sigma)U(\sigma), V\big)$$

(the second $\approx$ since $|R(\tau)U(\tau) - R(\sigma)U(\sigma)|$ is finite).

For (24) put $W = U(\tau)$ and then

$$\big((R(\tau) - R(\sigma))W, V\big) = \int_\sigma^\tau \left(\frac{dR}{d\eta} W, V\right) d\eta = -\int_\sigma^\tau \big(W\langle U(\eta), \nabla\rangle R, V\big)\, d\eta$$
$$= \int_\sigma^\tau \big(b\big(RU(\eta), W, V\big) + b\big(RU(\eta), V, W\big)\big)\, d\eta$$

using (3).

Since $\|W\| < \infty$, we have $\int_\tau^\sigma |b(RU(\eta), W, V)|^2\, d\eta$ and $\int_\tau^\sigma |b(RU(\eta), V, W)|^2\, d\eta$ both finite from (5) so the integrand on the right is S-integrable. Thus $((R(\tau) - R(\sigma))W, V) \approx 0$ when $\sigma \approx \tau$, as required. □

*3.4. Solving the stochastic nonhomogeneous Navier–Stokes equations*

Now we can prove the main existence theorem for nonhomogeneous stochastic Navier–Stokes equations.

**Theorem 3.6.** *Suppose that $u_0 \in \mathbf{H}$ and $\rho_0 \in L^\infty(D)$ with $0 < m \leqslant \rho_0(x) \leqslant M$, and $f, g$ satisfy Conditions* 3.2. *Then there is a weak solution $(\rho, u)$ to the stochastic nonhomogeneous Navier–Stokes equations with*

$$\mathbb{E}\left(\sup_{t \leqslant T} \big|u(t)\big|^2 + \nu \int_0^T \big\|u(t)\big\|^2\, dt\right) < E$$

*and for almost all $\omega$, for all $t$*

$$m \leqslant \rho(t, x) \leqslant M \quad \textit{for almost all } x.$$

**Proof.** Take $R_0 \in {}^*C^1(D)$ with $R_0 \approx \rho_0$ in the weak* topology: for example, since $C^1(D)$ is dense in $L^1(D)$ take $R_0 \approx \rho_0$ strongly in $L^1(D)$. Then $R_0(\xi) \approx {}^*\rho_0(\xi)$ for a.a. $\xi$, so by Loeb theory $\int (R_0 - {}^*\rho_0)\, {}^*z \approx 0$ for all $z \in L^1(D)$. We may assume that $m \leqslant R_0(\xi) \leqslant M$ for all $\xi$.

Let $(U(\tau), R(\tau))$ be the solution to the modified hyperfinite-dimensional Galerkin equations (18), (19) as defined in the previous section, with $U(0) = \mathrm{Pr}\, {}^*u(0)$ and $R(0) = R_0$, and let $\Omega_1$ be the full subset of $\Omega$ on which $\sup_{\tau \leqslant T} |U(\tau)|^2 + \nu \int_0^T \|U(\sigma)\|^2\, d\sigma$ is finite (so $U(\tau) = \overline{U}(\tau)$ all $\tau$) and the conclusions of Lemmas 3.4 and 3.5 hold.

**Definition of $u$.** The energy inequality (20) means that for all $(\tau, \omega)$ in a Loeb-full subset $\mathcal{E} \subseteq {}^*[0, T] \times \Omega$

$$\big\|U(\tau, \omega)\big\| < \infty,$$

so for each $(\tau, \omega) \in \mathcal{E}$ we have $U(\tau, \omega)$ strongly nearstandard in $\mathbf{H}$. The set

$$\mathcal{E}_0 = \big\{({}^\circ\tau, \omega):\ (\tau, \omega) \in \mathcal{E}\big\}$$

is a full subset of $[0,T]\times\Omega$. If $(\sigma,\omega),(\tau,\omega)\in\mathcal{E}$ with $\tau\approx\sigma$ then $U(\tau,\omega)\approx U(\sigma,\omega)$ in $\mathbf{H}$ and both are nearstandard in $\mathbf{H}$ (Lemma 3.5). So for $(t,\omega)\in\mathcal{E}_0$ define

$$u(t,\omega)={}^{\circ}U(\tau,\omega)$$

for any $(\tau,\omega)\in\mathcal{E}$ with $\tau\approx t$. Now for a.a. $\omega$ the set $\mathcal{E}_0(\omega)=\{t\colon (t,\omega)\in\mathcal{E}_0\}$ is a full subset of $[0,T]$ so for a.a. $\omega$ this defines $u(\cdot,\omega)$ as a process living in $\mathbf{H}$. It is clear that $u\in L^2([0,T]\times\Omega,\mathbf{V})$ and for a.a. $\omega$

$$u(\cdot,\omega)\in L^\infty(0,T;\mathbf{H})\cap L^2(0,T;\mathbf{V})$$

using (20) and $|{}^{\circ}U|\leqslant{}^{\circ}|U|$ and $\|{}^{\circ}U\|\leqslant{}^{\circ}\|U\|$. Let $\Omega_2$ be the full subset of $\Omega_1$ where this holds.

Note that $U$ is a lifting of $u$, and for $\omega\in\Omega_2$, $U(\cdot,\omega)$ is an $\mathrm{SL}^2$ lifting.

**Definition of $\rho$.** For a.a. $\omega$, we have $R(\tau,\omega)\in{}^*L^\infty(D)$ and $|R(\tau,\omega)|\leqslant M$ for all $\tau$ so we may immediately take standard parts (in the weak* topology): recall the notation introduced in Proposition 2.6 and define, for a.a. $\omega$

$$\widehat{R}(\cdot,\omega):{}^*[0,T]\to L^\infty(D)$$

by

$$\widehat{R}(\tau,\omega)=\widehat{R(\tau,\omega)}.$$

In order to obtain the function $\rho$ we show that for a.a. $\omega$, $\widehat{R}(\sigma,\omega)=\widehat{R}(\tau,\omega)$ whenever $\sigma\approx\tau$ (which is the same as saying that $R(\cdot,\omega)$ is weak* S-continuous).

Now for almost all $\omega$, $\sup_{\tau\leqslant T}|U(\tau,\omega)|$ is finite so $R(\cdot,\omega)$ obeys the density equation (17). Fix any such $\omega$. Take $z\in H^1(D)$; integration by parts (suppressing $\omega$) gives:

$$\frac{d}{d\tau}(R,{}^*z)=\left(\frac{dR}{d\tau},{}^*z\right)=\bigl(-\langle U,\nabla\rangle R,{}^*z\bigr)=\bigl(R,\langle U,\nabla\rangle{}^*z\bigr),$$

so for any $\tau\in[0,T]$

$$\bigl(R(\tau),{}^*z\bigr)=\bigl(R(0),{}^*z\bigr)+\int_0^\tau\bigl(R(\sigma),\bigl\langle U(\sigma),\nabla\bigr\rangle{}^*z\bigr)\,d\sigma.$$

Now the integrand $(R(\sigma),\langle(\sigma),\nabla\rangle\,{}^*z)$ is S-integrable since it is bounded (using (7)) and so for $\sigma\approx\tau$ we have $(R(\sigma),{}^*z)\approx(R(\tau),{}^*z)$ so that $(\widehat{R}(\sigma),z)=(\widehat{R}(\tau),z)$. Now $H^1(D)$ is dense in $L^1(D)$ so this shows that $\widehat{R}(\sigma)=\widehat{R}(\tau)$. Thus we may define, for all such $\omega$

$$\rho(t,\omega)=\widehat{R}(\tau,\omega)$$

for any $\tau\approx t$. Clearly $\rho\in L^\infty([0,T]\times D\times\Omega)$; in fact, $m\leqslant\rho(t,x,\omega)\leqslant M$ for a.a. $(t,x,\omega)$.

Thus $(\rho, u)$ satisfies conditions (i) and (ii) of Definition 3.1 and $\rho(0) = \widehat{R}_0 = \rho_0$. It remains to show that $(\rho, u)$ satisfy Eqs. (14), (15).

First note that there is a full subset $\mathcal{T}_0 \subseteq [0, T]$ such that if $t \in \mathcal{T}_0$ then the set $\widehat{\mathcal{E}_0}(t) = \{\omega\colon (t, \omega) \in \mathcal{E}_0\}$ is a full subset of $\Omega$. Consider a test function $\Phi \in C^1(0, T; \mathbf{V})$. We will show that for $T_0 \in \mathcal{T}_0$, for $\omega \in \widehat{\mathcal{E}_0}(t) \cap \Omega_2$ we have

$$\begin{aligned}&\big(\rho(T_0)u(T_0), \Phi(T_0)\big) - \big(\rho_0 u_0, \Phi(0)\big)\\ &\quad= \int_0^{T_0} \big[\big(\rho u, \Phi' + \langle u, \nabla\rangle\Phi\big) - \nu((u, \Phi)) + (\rho f, \Phi)\big]\, dt + \int_0^{T_0} (\Phi, \rho g)\, dw \end{aligned}\tag{25}$$

which is Eq. (14). It is clear that the internal pair $(R, U)$ satisfies the following

$$\begin{aligned}&\big(R(T_0)U(T_0), {}^*\Phi(T_0)\big) - \big(R_0 U(0), {}^*\Phi(0)\big)\\ &\quad= \int_0^{T_0} \big[\big(RU, {}^*\Phi'_\tau + \langle \overline{U}, \nabla\rangle {}^*\Phi\big) - \nu((U, {}^*\Phi)) + (RF, {}^*\Phi)\big]\, d\tau + \int_0^{T_0} ({}^*\Phi, R_N G\, dW_\tau). \end{aligned}\tag{26}$$

This is obtained from the internal equation

$$R_N(\tau)\left[\frac{dU}{d\tau} + \big\langle \overline{U}(\tau), \nabla\big\rangle U(\tau)\right] + \nu A U(\tau) = R_N(\tau) F\big(\tau, U(\tau)\big) + R_N G(\tau, U)\, dW_\tau$$

using integration by parts as in the proof of Lemma 3.4.

The procedure now to obtain (25) is to show that each of the terms in (26) has standard part equal to the corresponding term in (25). Fixing $T_0 \in \mathcal{T}_0$ and taking first the deterministic terms one by one, for $\omega \in \widehat{\mathcal{E}_0}(T_0) \cap \Omega_2$ observe that $\overline{U}(\tau, \omega) = U(\tau, \omega)$ for all $\tau$. Thus we have for such $\omega$ (suppressing $\omega$ where appropriate):

(1) From Corollary 2.8, since $U(0) \approx u_0$ and ${}^*\Phi(0) \approx \Phi(0)$ in $\mathbf{H}$ and $\widehat{R}_0 = \rho_0$ then

$$\big(R_0 U(0), {}^*\Phi(0)\big) \approx \big(\rho_0 u_0, \Phi(0)\big). \tag{27}$$

Take $\tau_0 \approx T_0$ with $(\tau_0, \omega) \in \mathcal{E}$ so that $\|U(\tau_0)\| < \infty$. Then $u(T_0) \approx U(\tau_0)$ in $\mathbf{H}$ and so again by Corollary 2.8, since $\Phi$ is continuous

$$\big(R(\tau_0)U(\tau_0), {}^*\Phi(\tau_0)\big) \approx \big(\rho(T_0)u(T_0), \Phi(T_0)\big). \tag{28}$$

(2) For a.a. $\tau$, $\|U(\tau)\| < \infty$ and so from Corollary 2.8,

$$\big(R(\tau)U(\tau), {}^*\Phi'(\tau)\big) \approx \big(\rho({}^\circ\tau)u({}^\circ\tau), \Phi'({}^\circ\tau)\big).$$

Moreover, $(R(\tau)U(\tau), {}^*\Phi'(\tau))$ is S-integrable since it is bounded, so we have

$$\int_0^{T_0} \big(R(\tau)U(\tau), {}^*\Phi'(\tau)\big)\, d\tau \approx \int_0^{T_0} {}^\circ\big(R(\tau)U(\tau), {}^*\Phi'(\tau)\big)\, d_L\tau$$

$$= \int_0^{T_0} \big(\rho({}^\circ\tau)u({}^\circ\tau), \Phi'({}^\circ\tau)\big)\, d_L\tau$$

$$= \int_0^{T_0} \big(\rho(t)u(t), \Phi'(t)\big)\, dt. \tag{29}$$

(3) Since $\|U(\tau)\| < \infty$ for a.a. $\tau$ we have for any such $\tau$, by Corollary 2.8

$$\begin{aligned}&\big(R(\tau)U(\tau), \langle U(\tau), \nabla\rangle {}^*\Phi(\tau)\big)\\ &\quad = {}^*b(RU, {}^*\Phi, U) \approx b\big(\rho({}^\circ\tau)u({}^\circ\tau), \Phi({}^\circ\tau), u({}^\circ\tau)\big) = \big(\rho({}^\circ\tau)u({}^\circ\tau), \langle u({}^\circ\tau), \nabla\rangle \Phi({}^\circ\tau)\big).\end{aligned}$$

To get the desired equality we need that the integrand ${}^*b(R(\tau)U(\tau), {}^*\Phi(\tau), U(\tau))$ is S-integrable. This follows from Lindstrøm's lemma and the following estimate, using (3):

$$\int_0^T \big|{}^*b\big(R(\tau)U(\tau), {}^*\Phi(\tau), U(\tau)\big)\big|^{\frac{4}{3}}\, d\tau \leqslant cM \int_0^T \big|U(\tau)\big|^{\frac{1}{2}} \big\|U(\tau)\big\|^2 \big\|{}^*\Phi(\tau)\big\|\, d\tau < \infty$$

on account of the energy bound (20). Hence

$$\begin{aligned}&\int_0^{T_0} \big(R(\tau)U(\tau), \langle U(\tau), \nabla\rangle {}^*\Phi(\tau)\big)\, d\tau\\ &\quad \approx \int_0^{T_0} {}^\circ\big(R(\tau)U(\tau), \langle U(\tau), \nabla\rangle {}^*\Phi(\tau)\big)\, d_L\tau\\ &\quad = \int_0^{T_0} \big(\rho({}^\circ\tau)u({}^\circ\tau), \langle u({}^\circ\tau), \nabla\rangle \Phi({}^\circ\tau)\big)\, d_L\tau = \int_0^{T_0} \big(\rho(t)u(t), \langle u(t), \nabla\rangle \Phi(t)\big)\, dt.\end{aligned} \tag{30}$$

(4) For almost all $\tau$ since $\|U(\tau)\| < \infty$ then $U(\tau) \approx {}^\circ U(\tau) = u({}^\circ\tau)$ weakly in $\mathbf{V}$ so

$$\big(\big(U(\tau), {}^*\Phi(\tau)\big)\big) \approx \big(\big(U(\tau), {}^*\Phi({}^\circ\tau)\big)\big) \approx \big(\big({}^\circ U(\tau), \Phi({}^\circ\tau)\big)\big) = \big(\big(u({}^\circ\tau), \Phi({}^\circ\tau)\big)\big)$$

for a.a. $\tau$. Moreover, $((U(\tau), {}^*\Phi(\tau)))$ is S-integrable because

$$\int_0^T \big(\big(U(\tau), {}^*\Phi(\tau)\big)\big)^2\, d\tau \leqslant \int_0^T \big\|U(\tau)\big\|^2\, d\tau \int_0^T \big\|{}^*\Phi(\tau)\big\|^2\, d\tau < \infty.$$

Thus

$$\begin{aligned}\int_0^{T_0} \big(\!\big(U(\tau), {}^*\Phi(\tau)\big)\!\big)\, d\tau &\approx \int_0^{T_0} {}^\circ\big(\!\big(U(\tau), {}^*\Phi(\tau)\big)\!\big)\, d_L\tau \\ &= \int_0^{T_0} \big(\!\big(u({}^\circ\tau), \Phi({}^\circ\tau)\big)\!\big)\, d_L\tau = \int_0^{T_0} \big(\!\big(u(t), \Phi(t)\big)\!\big)\, dt. \qquad (31)\end{aligned}$$

(5) Finally, consider the term involving $f$. We have, again using Corollary 2.8, that for almost all $\tau$

$$\big(R(\tau)F\big(\tau, U(\tau)\big), {}^*\Phi(\tau)\big) \approx \big(\rho({}^\circ\tau) f\big({}^\circ\tau, u({}^\circ\tau)\big), \Phi({}^\circ\tau)\big)$$

and it is routine to check that the integrand is S-integrable so that

$$\begin{aligned}\int_0^{T_0} \big(R(\tau){}^*f\big(\tau, U(\tau)\big), {}^*\Phi(\tau)\big)\, d\tau &\approx \int_0^{T_0} {}^\circ\big(R(\tau)F\big(\tau, U(\tau)\big), {}^*\Phi(\tau)\big)\, d_L\tau \\ &= \int_0^{T_0} \big(\rho({}^\circ\tau) f\big({}^\circ\tau, u({}^\circ\tau)\big), \Phi({}^\circ\tau)\big)\, d_L\tau \\ &= \int_0^{T_0} \big(\rho(t) f\big(t, u(t)\big), \Phi(t)\big)\, dt. \qquad (32)\end{aligned}$$

So for $\omega \in \widehat{\mathcal{E}}_0(T_0) \cap \Omega_2$ we have

$$\big(R(T_0)U(T_0), {}^*\Phi(T_0)\big) - \big(R_0 U(0), {}^*\Phi(0)\big) \approx \big(\rho(T_0)u(T_0), \Phi(T_0)\big) - \big(\rho_0 u_0, \Phi(0)\big) \quad \text{and}$$

$$\begin{aligned}&\int_0^{T_0} \big[\big(RU, {}^*\Phi'_\tau + \langle \overline{U}, \nabla\rangle {}^*\Phi\big) - \nu(\!(U, {}^*\Phi)\!) + (RF, {}^*\Phi)\big]\, d\tau \\ &\quad\approx \int_0^{T_0} \big[\big(\rho u, \Phi' + \langle u, \nabla\rangle \Phi\big) - \nu(\!(u, \Phi)\!) + (\rho f, \Phi)\big]\, dt.\end{aligned}$$

It remains to deal with the stochastic integral terms in (25), (26). For this we employ the theory of stochastic integration developed in [6,7]. This gives that for a.a. $\omega$

$$\int_0^{T_0} ({}^*\Phi, R_N G\, dW_\tau) \approx \int_0^{T_0} (\Phi, \rho g)\, dw \qquad (33)$$

provided that

(1) $\mathbb{E}(\int_0^{T_0} |R_N(\tau,\omega)G(\tau,U(\tau,\omega))|^2\,d\tau) < \infty$,
(2) for a.a. $(\tau,\omega)$

$$R_N(\tau,\omega)G\big(\tau,U(\tau,\omega)\big) \approx \rho({}^\circ\tau,\omega)_{\mathbf{H}}g\big({}^\circ\tau,u({}^\circ\tau,\omega)\big).$$

Now (1) follows from the energy bound (20) because

$$\big|R_N(\tau,\omega)G\big(\tau,U(\tau,\omega)\big)\big| \leqslant M\big|G\big(\tau,U(\tau,\omega)\big)\big| \leqslant M\,{}^*a(\tau)\big(1+\big|U(\tau)\big|\big).$$

For (2) we have that for a.a. $(\tau,\omega)$, $U(\tau,\omega)$ is near-standard and so

$$\begin{aligned} R_N(\tau,\omega)G\big(\tau,U(\tau,\omega)\big) &\approx \rho_{\mathbf{H}}({}^\circ\tau,\omega){}^\circ G\big(\tau,U(\tau,\omega)\big) \quad \text{(using Corollary 2.8)}\\ &= \rho_{\mathbf{H}}({}^\circ\tau,\omega)g\big({}^\circ\tau,{}^\circ U(\tau,\omega)\big) \quad \text{(by Anderson's Luzin theorem)}\\ &= \rho_{\mathbf{H}}({}^\circ\tau,\omega)g\big({}^\circ\tau,u({}^\circ\tau,\omega)\big) \end{aligned}$$

as required. This establishes that $(\rho,u)$ satisfies the stochastic equation (25) for the velocity field.

Turning now to the density equation (15), take $\varphi \in C^1(0,T;H^1(D))$ and any $T_0 \leqslant T$. For $\omega \in \Omega_2$ we have (since $\overline{U} = U$ here)

$$\frac{dR}{d\tau} + \big\langle U(\tau),\nabla\big\rangle R(\tau) = 0$$

and integration by parts shows that for a test function $\varphi \in C^1(0,T;H^1(D))$, for any $T_0 \leqslant T$,

$$\big(R(T_0),{}^*\varphi(T_0)\big) - \big(R_0,{}^*\varphi(0)\big) = \int_0^{T_0} \big(R,{}^*\varphi' + \langle U,\nabla\rangle{}^*\varphi\big)\,d\tau. \tag{34}$$

Now take standard parts of the terms in this equation one by one. We have

$$\big(R(T_0),{}^*\varphi(T_0)\big) \approx \big(\widehat{R}(T_0),\varphi(T_0)\big) = \big(\rho(T_0),\varphi(T_0)\big) \quad \text{and} \tag{35}$$

$$\big(R_0,{}^*\varphi(0)\big) \approx \big(\widehat{R}_0,\varphi(0)\big) = \big(\rho_0,\varphi(0)\big). \tag{36}$$

Also, for any $\tau$

$$\big(R(\tau),{}^*\varphi'(\tau)\big) \approx \big(\widehat{R}(\tau),\varphi'({}^\circ\tau)\big) = \big(\rho({}^\circ\tau),\varphi'({}^\circ\tau)\big).$$

Since $(R(\tau),{}^*\varphi'(\tau))$ is bounded it is S-integrable, so

$$\int_0^{T_0} \big(R(\tau),{}^*\varphi'(\tau)\big)\,d\tau \approx \int_0^{T_0} \big(\rho({}^\circ\tau),\varphi'({}^\circ\tau)\big)\,d_L\tau = \int_0^{T_0} \big(\rho(t),\varphi'(t)\big)\,dt. \tag{37}$$

Finally, the continuity property (7) shows that if $\|U(\tau)\| < \infty$ then

$$\big(R(\tau),\big\langle U(\tau),\nabla\big\rangle{}^*\varphi(\tau)\big) \approx \big(R(\tau),\big\langle{}^*u({}^\circ\tau),\nabla\big\rangle{}^*\varphi(\tau)\big) \approx \big(\widehat{R}({}^\circ\tau),\big\langle u({}^\circ\tau),\nabla\big\rangle\varphi({}^\circ\tau)\big)$$

using Proposition 2.6(i) with $z = \langle u({}^\circ\tau), \nabla\rangle\varphi({}^\circ\tau)$. Moreover, $(R(\tau), \langle U(\tau), \nabla\rangle {}^*\varphi(\tau))$ is S-integrable and so

$$\int_0^{T_0} \big(R(\tau), \big\langle U(\tau), \nabla\big\rangle^*\varphi(\tau)\big)\, d\tau \approx \int_0^{T_0} \big(\widehat{R}({}^\circ\tau), \big\langle u({}^\circ\tau), \nabla\big\rangle\varphi({}^\circ\tau)\big)\, d_L\tau$$
$$= \int_0^{T_0} \big(\rho(t), \big\langle u(t), \nabla\big\rangle\varphi(t)\big)\, dt. \tag{38}$$

Equations (35)–(38) show that for $\omega \in \Omega_2$, for any $T_0$

$$\big(\rho(T_0), \varphi(T_0)\big) - \big(\rho_0, \varphi(0)\big) = \int_0^{T_0} \big(\rho, \varphi' + \langle u, \nabla\rangle\varphi\big)\, dt$$

so the pair $(\rho, u)$ satisfies the random equation (15) for the density. This completes the proof of Theorem 3.6. □

### 3.5. Regularity in dimension 2

In the 2D setting there is more regularity to the solution, provided $g$ has a little more regularity.

**Theorem 3.7.** *Suppose that $d = 2$ and the initial condition $u_0 \in \mathbf{V}$ and $(\rho, u)$ is the solution to the stochastic nonhomogeneous Navier–Stokes equations constructed in the previous section. Suppose further that $g : [0, t] \times \mathbf{V} \to L(\mathbf{H}, \mathbf{V})$ and $|g(t, u)|_{\mathbf{H},\mathbf{V}} \leqslant a(t)(1 + \|u\|)$. Then almost surely*:

(a) $\sup_{t\in[0,T]} \|u(t)\| + \int_0^T |Au(t)|^2\, dt < \infty$;
(b) *$u(t)$ is strongly continuous in $\mathbf{H}$ and weakly continuous in $\mathbf{V}$*;
(c) *Equation* (14) *holds for* all $T_0 \leqslant T$.

**Proof.** The proof generalizes the idea of the proof of 2D-regularity in the homogeneous stochastic setting—see [7, Theorem 6.5.4 and Corollary 6.5.5]. For the sake of completeness we outline it here.

First recall the estimate

$$2\big|\big(\widetilde{R}_N^{-1} R_N \langle U, \nabla\rangle U, AU\big)\big| \leqslant 2c\frac{M}{m}|U|^{\frac{1}{2}}\|U\||AU|^{\frac{3}{2}} \leqslant c'|U|^2\|U\|^4 + \frac{\nu}{2M}|AU|^2 \tag{39}$$

obtained from (6) and Young's inequality. Take an auxiliary process

$$\xi(\tau) = \exp\left(-\int_0^\tau c'\big|U(\sigma)\big|^2\big\|U(\sigma)\big\|^2\, d\sigma\right) > 0 \quad \text{a.s.}$$

and note that $\xi(\tau)$ is decreasing, $\xi(0)=1$ and $\xi(T)\not\approx 0$ a.s. From the internal equation (18) Itô's lemma gives

$$\begin{aligned}
&\xi(\tau)\|U(\tau)\|^2\\
&\quad=\|U(0)\|^2+\int_0^\tau 2\xi(\sigma)((dU,U))+\int_0^\tau \xi(\sigma)\operatorname{tr}\bigl(Q_N A G^T G\bigr)\,d\sigma\\
&\qquad-c'\int_0^\tau \xi(\sigma)\bigl|U(\sigma)\bigr|^2\|U(\sigma)\|^4\,d\sigma\\
&\quad=\int_0^\tau \xi(\sigma)\bigl[-2\bigl(\widetilde{R}_N^{-1}R_N\langle U,\nabla\rangle U,AU\bigr)-2\nu\bigl(\widetilde{R}_N^{-1}AU,AU\bigr)+2\bigl(F(\tau,U),AU\bigr)\bigr]\,d\sigma\\
&\qquad+\int_0^\tau \xi(\sigma)\operatorname{tr}\bigl(Q_N A G^T G\bigr)\,d\sigma+2\xi(\sigma)((U,G))\,dW-c'\int_0^\tau \xi(\sigma)\bigl|U(\sigma)\bigr|^2\|U(\sigma)\|^4\,d\sigma.
\end{aligned}$$

Now use estimates (39) and

$$\begin{aligned}
\frac{1}{M}|AU|^2&\leqslant\bigl(\widetilde{R}_N^{-1}AU,AU\bigr),\\
2\bigl(F(\tau,U),AU\bigr)&\leqslant\frac{2M}{\nu}\bigl|F(\tau,U)\bigr|^2+\frac{\nu}{2M}|AU|^2\quad\text{(Young's inequality)},\\
\operatorname{tr}\bigl(Q_N A G^T G\bigr)&\leqslant\operatorname{tr}(Q_N)|G|^2_{\mathbf{H},\mathbf{V}}
\end{aligned}$$

and the growth conditions on $f$, $g$ together with $|U|^2\leqslant\lambda_1^{-1}\|U\|^2$. Then

$$\begin{aligned}
&\xi(\tau)\|U(\tau)\|^2-\|U(0)\|^2+\frac{\nu}{M}\int_0^\tau \xi(\sigma)|AU|^2\,d\sigma\\
&\quad\leqslant c''\int_0^\tau {}^*a^2(\sigma)\bigl(1+\xi(\sigma)\|U\|^2\bigr)\,d\sigma+\int_0^\tau 2\xi(\sigma)((U,G))\,dW
\end{aligned}$$

for a finite constants $c''$. From here the argument is exactly as in the proof of Theorem 6.5.4 of [7], using the Burkholder–Davis–Gundy inequality to estimate $\mathbb{E}\sup_{\tau\leqslant T}|M(\tau)|$ (where $M(\tau)$ is the martingale $\int_0^\tau \xi(\sigma)((U,G))\,dW$). This gives

$$\begin{aligned}
&\mathbb{E}\left(\sup_{\sigma\leqslant\tau}\xi(\sigma)\|U(\sigma)\|^2+\xi(\tau)\frac{\nu}{M}\int_0^\tau \bigl|AU(\sigma)\bigr|^2\,d\sigma\right)\\
&\quad\leqslant c'''\left(\|U(0)\|^2+\mathbb{E}\int_0^\tau {}^*a^2(\sigma)\bigl(1+\xi(\sigma)\|U(\sigma)\|^2\bigr)\,d\sigma\right)
\end{aligned}$$

for all $\tau$. Gronwall's lemma now gives the conclusion that

$$\mathbb{E}\left(\sup_{\tau\leqslant T}\xi(\tau)\left\|U(\tau)\right\|^2+\xi(T)\frac{\nu}{M}\int_0^T\left|AU(\tau)\right|^2d\tau\right)<\infty$$

and (a) follows using $\|{}^\circ U\|\leqslant{}^\circ\|U\|$ and $|{}^\circ AU|\leqslant{}^\circ|AU|$ and the fact that $\xi(T)\not\approx 0$ a.s.

For (b), we have almost surely that $\|U(\tau)\|$ is finite for all $\tau$. Then by Lemma 3.5 and the definition of $u$ we have $u({}^\circ\tau)={}^\circ U(\tau)$ for *all* $\tau$, with the standard part being strongly in $\mathbf{H}$. Lemma 3.5 also shows that $u$ is continuous strongly in $\mathbf{H}$ and weakly in $\mathbf{V}$.

For (c), referring to the proof of Theorem 3.6, if we take $\Omega_1$ to be the full set where $\sup_{\tau\leqslant T}\|U(\tau)\|^2+\int_0^T|AU(\tau)|^2d\tau$ is finite, then we may take $\mathcal{E}={}^*[0,T]\times\Omega_1$ so $\mathcal{E}_0=[0,T]\times\Omega_1$ and we may take $\mathcal{T}_0=[0,T]$. It follows from the proof of Theorem 3.6 that Eq. (14) then holds for all $T_0\leqslant T$. □

## 4. The deterministic nonhomogeneous Navier–Stokes equations

It is routine to modify (in fact simplify) the previous section to give a proof of existence (and regularity if $d=2$) for the deterministic nonhomogeneous incompressible Navier–Stokes equations that is somewhat more straightforward than the standard existence proofs. This results from setting $g=0$ throughout.

For the additional regularity when $d=2$, we can achieve a little more as in [2] using an alternative approach, essentially by translating the key ideas of [2] into our setting.

**Theorem 4.1.** *Suppose that $d=2$ and the initial condition $u_0\in\mathbf{V}$ and $(\rho(t),u(t))$ is the solution to the deterministic nonhomogeneous Navier–Stokes equations constructed by taking $g=0$ in the previous section. Then*

(a) $\sup_{t\in[0,T]}\|u(t)\|+\int_0^T|Au(t)|^2dt+\int_0^T|u_t(t)|^2dt<\infty$, *where $u_t$ denotes the time derivative $du/dt$*;
(b) *$u(t)$ is strongly continuous in $\mathbf{H}$ and weakly continuous in $\mathbf{V}$*;
(c) *for* all $T_0\leqslant T$, *for all $\Phi\in C^1(0,T;\mathbf{V})$*

$$\big(\rho(T_0)u(T_0),\Phi(T_0)\big)-\big(\rho_0u_0,\Phi(0)\big)=\int_0^{T_0}\big[\big(\rho u,\Phi'+\langle u,\nabla\rangle\Phi\big)-\nu((u,\Phi))+(\rho f,\Phi)\big]dt. \tag{40}$$

**Proof.** The internal equation for the Galerkin approximation $U(\tau)$ in $\mathbf{H}_N$ is

$$R_N(\tau)\left[\frac{dU}{d\tau}+\big\langle U(\tau),\nabla\big\rangle U(\tau)\right]+\nu AU(\tau)=R_N(\tau)F\big(\tau,U(\tau)\big) \tag{41}$$

and so

$$\frac{dU}{d\tau}=-\widetilde{R}_N^{-1}R_N\langle U,\nabla\rangle U-\nu\widetilde{R}_N^{-1}AU+F(\tau,U), \tag{42}$$

where we have dropped the dependence on $\tau$ where it is clear, to ease the notation. It follows that

$$\frac{d}{d\tau}\|U\|^2 = 2\left(\left(\frac{dU}{d\tau}, U\right)\right) = -2\big(\widetilde{R}_N^{-1} R_N \langle U, \nabla\rangle U, AU\big) - 2\nu\big(\widetilde{R}_N^{-1} AU, AU\big) + 2\big(F(\tau, U), AU\big)$$

and so

$$\begin{aligned}
\frac{d}{d\tau}\|U\|^2 + \frac{2\nu}{M}|AU|^2 &\leqslant \frac{d}{d\tau}\|U\|^2 + 2\nu\big(\widetilde{R}_N^{-1} AU, AU\big) \\
&= -2\big(\widetilde{R}_N^{-1} R_N \langle U, \nabla\rangle U, AU\big) + 2\big(F(\tau, U), AU\big) \\
&\leqslant 2c\frac{M}{m}|U|^{\frac{1}{2}}\|U\||AU|^{\frac{3}{2}} + \frac{2M}{\nu}\big|F(\tau, U)\big|^2 + \frac{\nu}{2M}|AU|^2 \\
&\leqslant c'|U|^2\|U\|^4 + \frac{\nu}{2M}|AU|^2 + \frac{2M}{\nu}\big|F(\tau, U)\big|^2 + \frac{\nu}{2M}|AU|^2
\end{aligned}$$

(we have used property (6) together with Young's inequality). The growth condition on $f$ together with $|U|^2 \leqslant \lambda_1^{-1}\|U\|^2$ gives finite constants $c', c''$ such that

$$\frac{d}{d\tau}\|U\|^2 + \frac{\nu}{M}|AU|^2 \leqslant c'|U|^2\|U\|^4 + c''\,{}^*a^2(\tau)\big(1 + \|U\|^2\big). \tag{43}$$

Now take an auxiliary function

$$\xi(\tau) = \exp\left(-\int_0^\tau c'\big|U(\sigma)\big|^2\big\|U(\sigma)\big\|^2\,d\sigma\right) > 0.$$

Elementary calculation gives

$$\begin{aligned}
\frac{d}{d\tau}\xi\|U\|^2 &= \xi\frac{d}{d\tau}\|U\|^2 + \|U\|^2\frac{d\xi}{d\tau} \\
&\leqslant \xi\big(c'|U|^2\|U\|^4 + c''\,{}^*a^2(\tau)\big(1 + \|U\|^2\big) - c'|U|^2\|U\|^4\big) \\
&\leqslant c''\,{}^*a^2(\tau)\big(1 + \xi\|U\|^2\big)
\end{aligned}$$

since $\xi(\tau) \leqslant 1$. Gronwall's lemma and the fact that $\xi(\tau)$ is decreasing and $\xi(T) \not\approx 0$ gives

$$\sup_{\tau\leqslant T}\big\|U(\tau)\big\|^2 \leqslant \xi(T)^{-1}\sup_{\tau\leqslant T}\xi(\tau)\big\|U(\tau)\big\|^2 < \infty.$$

Inserting this in (43) gives

$$\sup_{\tau\leqslant T}\big\|U(\tau)\big\|^2 + \int_0^T |AU|^2\,d\tau < \infty.$$

The results as stated in the theorem now follow easily: starting with (b), since $\|U(\tau)\|$ is finite for all $\tau$ then by Lemma 3.5 and the definition of $u$ we have $u({}^\circ\tau) = {}^\circ U(\tau)$ for *all* $\tau$, with the standard part being strongly in $\mathbf{H}$. Lemma 3.5 also shows that $u$ is continuous strongly in $\mathbf{H}$ and weakly in $\mathbf{V}$.

For (c), in the deterministic setting the set $\mathcal{T}_0$ in the proof of Theorem 3.6 is simply given by $\mathcal{T}_0 = \{{}^\circ\tau\colon \|U(\tau)\| < \infty\} = [0, T]$, and so the deterministic equation (40) holds for all $T_0 \leqslant T$.

Finally, since $\|{}^\circ U\| \leqslant {}^\circ\|U\|$ and $|{}^\circ AU| \leqslant {}^\circ|AU|$ and $Au = {}^\circ AU$ whenever $|AU|$ is finite, then the first two terms of (a) are finite. For the derivative of $u$, we have from (42)

$$U(t) = U(0) + \int_0^t \bigl[-\widetilde{R}_N^{-1} R_N \langle U, \nabla\rangle U - \nu \widetilde{R}_N^{-1} AU + F(\tau, U)\bigr]\, d\tau = U(0) + \int_0^t \Psi(\tau)\, d\tau.$$

Now check that $|\Psi(\tau)|$ is S-integrable; this follows from

$$\bigl|\widetilde{R}_N^{-1} R_N \langle U, \nabla\rangle U\bigr| \leqslant \frac{cM}{m} |U|^{\frac{1}{2}} \|U\| |AU|^{\frac{1}{2}} \quad \text{and} \quad \bigl|\widetilde{R}_N^{-1} AU\bigr| \leqslant M^{-1}|AU|$$

and the growth condition on $f$. It is also clear that $|\Psi(\tau)|$ is finite for a.a. $\tau$ so we have

$$u(t) = u(0) + \int_0^t {}^\circ\Psi(\tau)\, d_L\tau = u(0) + \int_0^t \psi(\tau)\, d_L\tau,$$

where $\psi = \mathbb{E}({}^\circ\Psi | \mathrm{st}^{-1}\mathfrak{B})$ with $\mathfrak{B} =$ the Borel algebra on $[0, T]$. So $u_t$ is given by $u_t({}^\circ\tau) = \psi(\tau)$ and we have

$$\int_0^T |u_t(t)|^2\, dt = \int_0^t \bigl|\psi(\tau)\bigr|^2\, d_L\tau \leqslant \int_0^t \bigl|{}^\circ\Psi(\tau)\bigr|^2\, d_L\tau \leqslant {}^\circ\int_0^t \bigl|\Psi(\tau)\bigr|^2\, d\tau < \infty. \qquad \square$$

## Appendix A. Nonstandard analysis

We provide here a brief outline of the basics of nonstandard analysis and Loeb measure theory as employed in this paper. For full details see any of the modern expositions of this methodology—for example [1,7,11] or [12].

### A.1. The nonstandard universe

The starting point is a base set $\mathbb{B}$ that contains all the standard objects involved in our discussion. In particular, $\mathbb{B}$ should contain the set of reals $\mathbb{R}$ and the linear space $\mathbf{H}$. The following *superstructure over* $\mathbb{B}$, denoted by $\mathbb{V} = V(\mathbb{B})$, is then an adequate (standard) mathematical universe in which to treat the stochastic Navier–Stokes equations[3] (where $\mathcal{P}(A)$ denotes the set of all subsets of a set $A$):

[3] That is, all objects discussed and constructed live in this collection of sets.

$$V_0(\mathbb{B}) = \mathbb{B}, \qquad V_{n+1}(\mathbb{B}) = V_n(\mathbb{B}) \cup \mathcal{P}\big(V_n(\mathbb{B})\big), \quad n \in \mathbb{N} \quad \text{and}$$
$$\mathbb{V} = V(\mathbb{B}) = \bigcup_{n\in\mathbb{N}} V_n(\mathbb{B}).$$

The an ultrapower construction is used to build a *nonstandard extension* ${}^*\mathbb{B} \supset \mathbb{B}$, and at the same time a mapping ${}^*: V(\mathbb{B}) \to V({}^*\mathbb{B})$ that associates to each set $A \in \mathbb{V}$ a nonstandard counterpart ${}^*A \in V({}^*\mathbb{B})$. At level 0, we simply have ${}^*b = b$ for each $b \in \mathbb{B}$. At level 1, for each $A \subset \mathbb{B}$ we have $A \subset {}^*A \subset {}^*\mathbb{B}$, with ${}^*A \setminus A$ consisting of "ideal" or "nonstandard" elements. For example, ${}^*\mathbb{N} \setminus \mathbb{N}$ consists of infinite (hyper)natural numbers.

In general, for each set $A \in \mathbb{V}$, the mapping ${}^*$ maps $A$ injectively into ${}^*A$. So even for mathematical objects[4] $\mathcal{J}$ at higher levels, ${}^*\mathcal{J}$ can be regarded as an extension of $\mathcal{J}$.

The resulting *nonstandard universe* is the collection

$${}^*\mathbb{V} = \{x\colon\ x \in {}^*A \text{ for some } A \in \mathbb{V}\}$$

consisting of all members of nonstandard counterparts of sets in $\mathbb{V}$. Although ${}^*\mathbb{V} \subset V({}^*\mathbb{B})$, it is crucial to realize that ${}^*\mathbb{V}$ is *not* the same as $V({}^*\mathbb{B})$. Sets in ${}^*\mathbb{V}$ are known as *internal sets*; a set is *external* if it is not internal. Of course, all sets (including those in $V({}^*\mathbb{B})$) are ordinary sets in the usual universe of mathematics.

The key property of the nonstandard universe that makes it tractable is the *transfer principle*, which indicates precisely which properties of the superstructure $\mathbb{V}$ are inherited by ${}^*\mathbb{V}$.

**Theorem A.1** *(The transfer principle). Suppose that $\varphi$ is a bounded quantifier statement. Then $\varphi$ holds in $\mathbb{V}$ if and only if ${}^*\varphi$ holds in ${}^*\mathbb{V}$.*

A *bounded quantifier statement* (bqs) is a statement of mathematics that can be written in such a way that all quantifiers range over a prescribed set. That is, subclauses such as $\forall x \in A$ and $\exists y \in B$ are permitted but not unbounded quantifiers such as $\forall x$ and $\exists y$. Most quantifiers in mathematical practice are bounded (often only implicitly in exposition). A bqs $\varphi$ may also contain fixed sets $\mathcal{M}$ from $\mathbb{V}$, which will be replaced in ${}^*\varphi$ by ${}^*\mathcal{M}$.

Members of internal sets are internal (this follows easily from the construction) and since the sets ${}^*\mathcal{M}$ are also internal, it follows that the information obtained from the transfer principle is entirely about *internal sets.*

It is possible (and quite convenient) to take an axiomatic approach to ${}^*\mathbb{V}$, which simply postulates the existence of a set ${}^*\mathbb{V}$ and a mapping ${}^*: \mathbb{V} \to {}^*\mathbb{V}$ that obeys the transfer principle. For most purposes (and certainly the construction of Loeb measures) the further assumption of $\aleph_1$-*saturation* is needed—a property that comes with the ultrapower construction.

*A.1.1. $\aleph_1$-saturation*

**Definition A.2.** A nonstandard universe ${}^*\mathbb{V}$ is said to be $\aleph_1$-*saturated* if the following holds:

> if $(A_m)_{m\in\mathbb{N}}$ is a countable decreasing sequence of *internal* sets with each $A_m \neq \emptyset$, then $\bigcap_{m\in\mathbb{N}} A_m \neq \emptyset$.

[4] We are taking the approach that every mathematical object is actually a set.

**Theorem A.3.** *A nonstandard universe* ${}^*\mathbb{V}$ *constructed as a countable ultrapower is* $\aleph_1$*-saturated.*

$\aleph_1$-saturation is a kind of compactness property that is essential for the Loeb measure construction.

*A.1.2. The hyperreals* ${}^*\mathbb{R}$

The nonstandard extension ${}^*\mathbb{R}$ of the reals is called the *hyperreals* and the following are fundamental definitions in connection with these. Let $x, y \in {}^*\mathbb{R}$. Then

**Definition A.4.**

(i) $x$ is *finite* if $|x| < n$ for some $n \in \mathbb{N}$; $x$ is *infinite* if not finite;
(ii) $x$ is *infinitesimal* if $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$;
(iii) $x \approx y$ if $|x - y|$ is infinitesimal (so $x \approx 0$ iff $x$ is infinitesimal).

The following is essentially a reflection of the fact that in $\mathbb{R}$ bounded closed sets are compact.

**Theorem A.5** *(Standard part theorem). If* $x \in {}^*\mathbb{R}$ *is finite then there is a unique* $r \in \mathbb{R}$ (*called the* standard part of $x$) *such that* $x \approx r$.

*A.2. Further nonstandard preliminaries*

*A.2.1. Topology*

Given a standard Hausdorff space $S$, we identify each point $x \in S$ with ${}^*x \in {}^*S$, so that $S \subseteq {}^*S$.[5] If $x \in S$ and $X \in {}^*S$, then $x$ is the *standard part of* $X$, in symbols $x = {}^\circ X$, if $X \in {}^*O$ for every open neighborhood $O$ of $x$. Since $S$ is Hausdorff, each $X \in {}^*S$ has at most one standard part. An element $X \in {}^*S$ is said to be *nearstandard*, in symbols $X \in \mathrm{ns}(S)$, if $X$ has a standard part. Thus the standard part function maps $\mathrm{ns}(S)$ onto $S$ and is the identity on $S$. The *standard part* of a set $B \subseteq \mathrm{ns}(S)$ is the set ${}^\circ B = \mathrm{st}(B) = \{{}^\circ X\colon\ X \in B\}$.

In the particular case of a standard metric space $(S, \rho)$ (and *a fortiori* a Hilbert space such as occurs in the discussion of the paper), $x = {}^\circ X$ if and only if ${}^*\rho(X, x) \approx 0$, and two points $X, Y \in {}^*S$ are said to be *infinitely close*, in symbols $X \approx Y$, if ${}^*\rho(X, Y) \approx 0$. This is consistent with the notation introduced earlier for ${}^*\mathbb{R}$.

In the case of a Banach space $S$ with dual $S'$ note the following characterization of the weak and weak* topologies.

**Lemma A.6.** *Let* $S$ *be a Banach space.*

(a) *If* $x \in S$ *and* $X \in {}^*S$ *then* $x \approx X$ *in the weak topology of* $S$ *iff* $y(x) \approx {}^*y(X)$ *for all* $y \in S'$.
(b) *If* $y \in S'$ *and* $Y \in {}^*S'$ *then* $y \approx Y$ *in the weak* topology of* $S'$ *iff* $y(x) \approx Y(x)$ *for all* $x \in S$.

Note the following well-known compactness criterion that is needed.

**Proposition A.7.** *Let* $S$ *be a separable space, and* $C \subseteq S$. *Then the following are equivalent*:

(i) *$C$ is compact,*
(ii) ${}^*C \subseteq \mathrm{ns}(C)$ (*that is, each* $x \in {}^*C$ *is nearstandard and* ${}^\circ x \in C$).

[5] If $S \subset \mathbb{B}$ then we actually have $x = {}^*x$.

*A.2.2. Nonstandard representation of the space* $\mathbf{H}$

The space ${}^*\mathbf{H}$ has a basis $({}^*e_n)_{n\in{}^*\mathbb{N}} = (E_n)_{n\in{}^*\mathbb{N}}$ given by the nonstandard extension ${}^*e$ of the function $e:\mathbb{N}\to\mathbf{H}$. For each $U\in{}^*\mathbf{H}$ there is a unique internal sequence of hyperreals $(U_n)_{n\in{}^*\mathbb{N}}$ such that

$$U=\sum_{n=1}^{{}^*\infty} U_n E_n$$

and the subspace $\mathbf{H}_N$ of ${}^*\mathbf{H}$ is defined by

$$\mathbf{H}_N=\left\{\sum_{n=1}^{N} U_n E_n\colon\ U\in{}^*\mathbb{R}^N,\ U \text{ internal}\right\}.$$

On $\mathbf{H}$ (and also $\mathbf{V}$ or any of the spaces $\mathbf{H}^r$) there are both weak and strong topologies. The book [7] gives full information about the nonstandard characterizations of these, and other matters, but here are the most important facts.

**Proposition A.8.** *Let $U\in{}^*\mathbf{H}$ (or $\mathbf{H}_N$). Then*

(a) *$U$ is (strongly) nearstandard to $u$ in $\mathbf{H}$ (denoted $U\approx u$) if $|U-{}^*u|\approx 0$ (and then $|U|\approx|u|$).*
(b) *$U$ is weakly nearstandard to $u$ in $\mathbf{H}$ (denoted $U\approx_w u$) if $(U,{}^*v)\approx(u,v)$ for all $v\in\mathbf{H}$ and $|u|\leqslant{}^\circ|U|$ (allowing this to be $\infty$).*
(c) *If $U$ is strongly nearstandard then it is weakly nearstandard and the standard parts agree, so we write ${}^\circ U$ for the standard part in whichever topology it may be nearstandard.*
(d) *If $|U|$ is finite then $U$ is weakly nearstandard and the weak standard part $u=\mathrm{st}_{\mathrm{weak}}(U)$ is defined by*

$$u_n={}^\circ(U_n)$$

*for finite $n$.*
(e) *If $\|U\|$ is finite then $U$ is strongly nearstandard in $\mathbf{H}$.*
(f) *If $|U|,|V|<\infty$ then*

$$U\approx_w V\quad\Longleftrightarrow\quad U_i\approx V_i\quad\forall i\in\mathbb{N}.$$

Similar facts obtain for other spaces in the spectrum $\mathbf{H}^r$.

*A.2.3. Loeb measure theory*

A Loeb measure is a *standard* measure that is constructed from an internal measure space, as follows. Suppose that $\mathbf{X}=(X,\mathcal{A},M)$ is a finitely additive internal space so that $\mathcal{A}$ is an internal algebra of subsets of the set $X$ and $M:\mathcal{A}\to{}^*\mathbb{R}^{\geqslant 0}$ is finitely additive, and suppose that $M(X)$ is finite. The *Loeb measure* is the unique extension to the $\sigma$-algebra $\sigma(\mathcal{A})$ of the function $({}^\circ M)(A)={}^\circ(M(A))$. After completion (that is, including all null-sets) this gives a standard complete measure space

$$\mathbf{X}_L=\big(X,L(\mathcal{A}),M_L\big)$$

which is the *Loeb measure space* generated by $(X,\mathcal{A},M)$. The completion $L(\mathcal{A})$ is the set of *Loeb measurable* subsets of $X$, which can be characterized as those sets that are approximable from below and above by internal sets from $\mathcal{A}$. The important points to note are: (a) this is a *standard* space—$L(\mathcal{A})$ is a $\sigma$-algebra extending $\sigma(\mathcal{A})$, and the $\sigma$-additive set function $M_L$ that extends ${}^\circ M$ is real valued (that is, $M_L : L(\mathcal{A}) \to \mathbb{R}^{\geqslant 0}$), (b) the domain of this measure is the original set $X$, which happens to be a set in the nonstandard universe, but is still a *bona fide* set.

An important example of a Loeb measure is the *Loeb–Lebesgue* measure $\Lambda_L = {}^*\lambda_L$ on an interval ${}^*[0, T]$, where $\lambda$ is Lebesgue measure. The Lebesgue measure can be recovered from this by the following result.

**Proposition A.9.** *Let $B \subseteq [0, T]$. Then*

(a) *$B$ is Lebesgue measurable iff and only* $\mathrm{st}^{-1}(B) = \{x \in {}^*[0, T]: {}^\circ x \in B\}$ *is Loeb measurable.*
(b) *If $B$ is Lebesgue measurable then $\lambda(B) = \Lambda_L(\mathrm{st}^{-1}(B))$.*

An application of this is the following lemma.

**Lemma A.10.** *If $A$ is a full subset of* ${}^*[0, T]$ *then* $\mathrm{st}(A)$ *is a Lebesgue full subset of* $[0, T]$.

**Proof.** Let $B = \mathrm{st}(A)$. Then $A \subseteq \mathrm{st}^{-1}(B) \subseteq {}^*[0, T]$ so $\mathrm{st}^{-1}(B)$ is Loeb measurable with full measure. By the result above $B$ is Lebesgue measurable and $\lambda(B) = \Lambda_L(\mathrm{st}^{-1}(B)) = T$. □

**Loeb integration.** Loeb integration theory gives the connection between internal integration using the internal measure $M$ and standard integration in the Loeb space. For finitely bounded internal functions $F : M \to {}^*\mathbb{R}$ this is simply

$$\int_X F\,dM \approx \int_X {}^\circ F\,dM_L. \tag{A.1}$$

This relationship extends to a larger class of internal functions $F : M \to {}^*\mathbb{R}$ that are called *S-integrable*, defined by the property that $\int_X |F|\,dM$ is finite and $\int_A |F|\,dM \approx 0$ whenever $A \in \mathcal{A}$ with $M(A) \approx 0$.

We note, finally, the following fact that is needed, remembering that $\mathbf{H} \subset \mathbf{L}^2(D)$.

**Proposition A.11.** *Let $U \in \mathbf{H}_N$ and $u \in \mathbf{H}$. Then the following are equivalent*:

(a) *$U \approx u$ in the strong* (*i.e. metric*) *topology of* $\mathbf{H}$.
(b) *$U$ is an* $\mathrm{SL}^2$ lifting *of $u$*; *that is, $U^2 : {}^*D \to {}^*\mathbb{R}$ is S-integrable and for almost all* (*with respect to the Loeb–Lebesgue measure*) *$x \in {}^*D$*

$$U(x) \approx u({}^\circ x).$$

**Stochastic integration.** There is a well-developed theory of stochastic integration paralleling the above integration theory. The basic idea is that if an internal probability space $\boldsymbol{\Omega}$ carries an internal martingale $W$ say for which an internal stochastic integral $\int_0^t F\,dW$ is defined (where

$F : {}^*[0, T] \times \Omega \to {}^*\mathbb{R}$) then under appropriate conditions we have

$$\int_0^t F\,dW \approx \int_0^t {}^\circ F\,dw$$

for Loeb-almost-all $\omega \in \Omega$, where $w = {}^\circ W$ is a martingale. The original theory was developed by Anderson in finite-dimensional spaces with $W$ an internal infinitesimal random walk, but it extends quite naturally to the present infinite-dimensional context with $W$ an internal hyperfinite-dimensional Wiener process whose standard part ${}^\circ W$ is a Wiener process on the Hilbert space $\mathbf{H}$. For full details consult [7].

## References


[1] S. Albeverio, J.-E. Fenstad, R. Høegh-Krohn, T.L. Lindstrøm, Nonstandard Methods in Stochastic Analysis and Mathematical Physics, Academic Press, New York, 1986.

[2] S.N. Antonsev, A.V. Kazhikhov, V.N. Monakhov, Boundary Value Problems in Mechanisms of Nonhomogeneous Fluids, Elsevier Science, Amsterdam, 1990; English translation of: Kraevye Zadachi Mekhaniki Neodnorodnykh Zhidkostei, Nauka Publishers, Novosibirsk, 1983 (in Russian).

[3] H. Bessaih, Semi-linearized compressible Navier–Stokes equations perturbed by noise, Electron. J. Differential Equations 2003 (2) (2003), 18 pp.

[4] J.L. Boldrini, M.A. Rojas-Medar, E. Fernández-Cara, Semi-Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids, J. Math. Pures Appl. 82 (2003) 1499–1525.

[5] M. Capiński, N.J. Cutland, A simple proof of existence of weak and statistical solutions of Navier–Stokes equations, Proc. R. Soc. Lond. Ser. A 436 (1992) 1–11.

[6] M. Capiński, N.J. Cutland, Stochastic Navier–Stokes equations, Applicandae Math. 25 (1991) 59–85.

[7] M. Capiński, N.J. Cutland, Nonstandard Methods for Stochastic Fluid Mechanics, Ser. Adv. Math. Appl. Sci., vol. 27, World Scientific, Singapore, 1995.

[8] M. Capiński, N.J. Cutland, Existence of global stochastic flow and attractors for Navier–Stokes equations, Probab. Theory Related Fields 115 (1999) 121–151.

[9] Y. Cho, H. Kim, Unique solvability for the density-dependent Navier–Stokes equations, Nonlinear Anal. 59 (2004) 465–489.

[10] H.J. Choe, H. Kim, Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids, Comm. Partial Differential Equations 28 (2003) 1183–1201.

[11] N.J. Cutland, Loeb measure theory, in: N.J. Cutland, F. Oliveira, V. Neves, J. Sousa-Pinto (Eds.), Developments in Nonstandard Mathematics, in: Pitman Res. Notes Math., vol. 336, Longman Sci. Tech., Harlow, 1995, pp. 151–177.

[12] N.J. Cutland, Loeb Measures in Practice—Recent Advances, Lecture Notes in Math., vol. 1751, Springer, Berlin, 2000, vii+111 pp.

[13] N.J. Cutland, K. Grzesiak, Optimal control for 2D stochastic Navier–Stokes equations, Appl. Math. Optim., in press.

[14] N.J. Cutland, K. Grzesiak, Optimal control for 3D stochastic Navier–Stokes equations, Stochastics 77 (5) (2005) 437–454.

[15] N.J. Cutland, H.J. Keisler, Global attractors for 3D stochastic Navier–Stokes equations, J. Dynam. Differential Equations 16 (2004) 205–266.

[16] N.J. Cutland, H.J. Keisler, Attractors and neo-attractors for 3D stochastic Navier–Stokes equations, Stoch. Dyn. 5 (4) (2005) 487–533.

[17] B.E. Enright, A nonstandard approach to the stochastic nonhomogeneous Navier–Stokes equations, PhD thesis, University of Hull, UK, 1999.

[18] A. Ichikawa, Stability of semilinear stochastic evolution equations, J. Math. Anal. Appl. 90 (1982) 12–44.

[19] A.V. Kazhikhov, Solvability of the initial and boundary-value problem for the equations of motion of an inhomogeneous viscous incompressible fluid, Soviet Phys. Dokl. 19 (6) (1974) 331–332; English translation of the paper in: Dokl. Akad. Nauk SSSR 216 (6) (1974) 1240–1243.

[20] J.U. Kim, Weak solutions of an initial boundary value problem for an incompressible viscous fluid with nonnegative density, SIAM J. Math. Anal. 18 (1) (1987) 89–96.

[21] P.-L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1. Incompressible Models, Oxford Lecture Ser. Math. Appl., vol. 10, Oxford Univ. Press, Oxford, 1996.
[22] R. Salvi, The equations of viscous incompressible nonhomogeneous fluids: On the existence and regularity, J. Aust. Math. Soc. Ser. B 33 (1991) 94–110.
[23] J. Simon, Ecoulement d'un fluide non homogène avec une densité initiale s'annulant, C. R. Acad. Sci. Paris Ser. A 15 (1978) 1009–1012.
[24] J. Simon, Nonhomogeneous viscous incompressible fluids: Existence of velocity, density, and pressure, SIAM J. Math. Anal. 21 (5) (1990) 1093–1117.
[25] R. Temam, Navier–Stokes Equations and Nonlinear Functional Analysis, SIAM, Philadelphia, PA, 1983.
[26] E. Tornatore, Global solution of bi-dimensional stochastic equation for a viscous gas, Nonlinear Differential Equations Appl. 7 (4) (2000) 343–360.
[27] H.F. Yashima, Equations de Navier–Stokes stochastiques non homogènes et applications, Thesis, Scuola Normale Superiore, Pisa, 1992.
[28] H.F. Yashima, Equations stochastiques d'un gaz visqueux isotherme dans un domaine monodimensionnel infini, Acta Math. Vietnam. 26 (2) (2001) 147–168.